H. A. Cohen
21 Dicembre 1967
Il Nuovo Cimento
Serie X, Vol. 52 A, pag. 1242-1253

Covariant Wave Equations for Charged Particles of Higher Spin in an Arbitrary Gravitational Field.

H. A. Cohen

Department of Mathematical Physics, University of Adelaide - Adelaide

(ricevuto il 22 Novembre 1966; manoscritto revisionato ricevuto il 12 Giugno 1967)

Summary. It is known that special difficulties are encountered in devising a wave equation to describe higher spin $S \geqslant \frac{3}{2}$ particles in interaction with the electromagnetic field, and Buchdahl has shown that like difficulties arise when the gravitational field is introduced. We show that just as a consistent electromagnetic interaction can be introduced, so wave equations describing charged particles of spin $S = \frac{3}{2}$ and of spin $S = 2$ in a universe endowed with a Riemannian metric can be devised. The paper incorporates an account of Dirac γ matrices in general relativity.

1. – Introduction.

The problem of devising wave equations appropriate to charged particles of higher spin $S \geqslant \frac{3}{2}$ in an arbitrary universe endowed with a Riemannian metric is an extension of the problem of devising the flat-space wave equation for the presence of an electromagnetic field. The difficulties in both cases are of the same nature: a free particle of higher spin S, in flat space, is described by a wave function which satisfies some field equation together with a number of supplementary conditions, so that the wave function has exactly $2S+1$ (boson) or $2(2S+1)$ (fermion) degrees of freedom [1]; on making a «minimal» extension, to introduce a gravitational or electromagnetic interaction, one finds, for $S \geqslant \frac{3}{2}$, that the modified supplementary condition will combine with the

[1] P. A. M. Dirac: *Proc. Roy. Soc.*, A **155**, 447 (1936); M. Fierz: *Helv. Phys. Acta*, **12**, 3 (1938).

9062

generalized wave equation to yield further constraint equations, so that two possibilities arise. Either the wave function will be found to no longer have the number of independent components necessary to describe a spin-S particle, or the particle wave function will factor out from the new constraints, which will thus restrict other fields. These two possibilities are not always distinct, as shown in our discussion of the spin-$\frac{3}{2}$ wave equation in Sect. **3**.

Just as the problem of introducing an electromagnetic interaction for higher-spin particles was solved, in the first place by FIERZ and PAULI ([2]), likewise the more general problem is amenable to solution by methods which are a natural generalization of the methods used to solve the simpler problem.

In Sect. **2** a brief account of Dirac spinors in general relativity is presented, attention being focused on what we term spectors—quantities with both vector and Dirac spinor indices, an example being the Rarita-Schwinger ([3]) wave function ψ_μ. In Sect. **3** we show that the Rarita-Schwinger equations, on « minimal » extension, are satisfactory to describe spin-$\frac{3}{2}$ particles only in the absence of electromagnetic interaction and then only if the space is of constant Riemannian curvature. This result is in agreement with an investigation carried out by BUCHDAHL ([4]) using 2-component-spinor wave functions.

In Sect. **4** we extend an elegant solution of the electromagnetic interaction problem given by MOLDAUER and CASE ([5]) to find covariant spector wave equations to adequately describe spin-$\frac{3}{2}$. The same approach is made to the derivation of the spin-2 wave equation, which is presented in Sect. **5**.

Barring unforseen difficulties, we anticipate that the methods that we use can be applied successfully to the general problem of devising covariant wave equations for charged particles of spin $S > 2$; however, the computations involved will become exceedingly lengthy.

JOHNSON and SUDARSHAN ([6]) have proved that the Lorentz covariant theory of charged spin-$\frac{3}{2}$ particles cannot be quantized if $F_{ij} \neq 0$. It is of interest to ask if in the absence of an electromagnetic field, but in the presence of a gravitational field, our theory of spin $\frac{3}{2}$ can be quantized; we do not here investigate this question.

([2]) M. FIERZ and W. PAULI: *Proc. Roy. Soc.*, A **173**, 211 (1939); a matrix formulation of the spin-$\frac{3}{2}$ theory as extended to involve auxiliary spinors is given by S. N. GUYPTA: *Phys. Rev.*, **95**, 1334 (1954).

([3]) W. RARITA and J. SCHWINGER: *Phys. Rev.*, **60**, 61 (1941).

([4]) H. A. BUCHDAHL: *Nuovo Cimento*, **10**, 96 (1958); **25**, 486 (1962).

([5]) P. A. MOLDAUER and K. M. CASE: *Phys. Rev.*, **102**, 280 (1956). The account given of the spin-$\frac{3}{2}$ wave equation is correct, but for fermions of spin $S > \frac{3}{2}$ these authors postulated symmetry conditions for the wave function which are inconsistent with the wave equation.

([6]) K. JOHNSON and E. C. G. SUDARSHAN: *Ann. of Phys.*, **13**, 126 (1961).

9063

2. – Spectors in general relativity.

A covariant quantity with both vector and Dirac spinor idices we shall refer to as a spector: one example of a spector is given by the set of four γ_μ matrices, each matrix having one (normal) Dirac index and one adjoint Dirac index; another example is provided by the Rarita-Schwinger (3) wave function ψ_μ introduced later in this Section. The aim of this Section is to briefly expound the properties of spectors relevant to our purpose.

In a classic paper INFELD and VAN DER WAERDEN (7) provided the definitive account of 2-component spinors in general relativity; the approach of these authors could be applied to the parallel problem of Dirac spinors (*). However, we prefer to follow GREEN (8) and adopt the attractive view-point that the fundamental geometric quantity is the spector γ_μ, *i.e.* the set of four Dirac matrices, the metric tensor $g_{\mu\nu}$ being defined by the relations

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,, \tag{2.1}$$

$$[g_{\mu\nu}, \gamma_\lambda] = 0\,. \tag{2.2}$$

Just as is the case in special relativity, the ring generated by the γ_λ has just 16 independent elements. One defines the index raising operator $g^{\lambda\mu}$ by

$$g^{\lambda\mu}g_{\mu\nu} = \delta_\nu^{\ \lambda}\,, \tag{2.3}$$

so that $\gamma^\mu = g^{\lambda\mu}\gamma_\lambda$ etc. A Dirac spinor ψ is specified to be a column matrix acted on the left by the matrices of the γ_μ ring, while we use a tilde (~) to label a Dirac adjoint spinor $\tilde{\psi}$. It is postulated that the covariant derivative of a Dirac spinor is given in terms of the spinor affinity Γ_μ by

$$\psi_{;\mu} = \psi_{,\mu} - \Gamma_\mu\psi\,; \qquad \tilde{\psi}_\mu = \tilde{\psi}_{,\mu} + \tilde{\psi}\Gamma_{\mu,}\,, \tag{2.4}$$

where semi-colon denotes covariant derivative, comma denotes ordinary derivative. The spinor affinity Γ_μ thus must belong to the γ_μ ring, and is restrained

---

(7) L. INFELD and B. L. VAN DER WAERDEN: *Sitz. Ber. d. Preuss. Akad. d. Wiss.*, **9**, 380 (1933).

(*) In the paper cited, INFELD and VAN DER WAERDEN consider only special representations of the generalized Dirac matrices.

(8) H. S. GREEN: *Nucl. Phys.*, **7**, 373 (1958). This paper contains earlier references. to the rather limited use that has been made of the Dirac γ matrices in general relativity. Green's interesting findings as to the dimensions of the irreducible representations of the γ matrices are irrelevant to the present purpose.

by the requirement that

$$\gamma_{\lambda;\mu} = \gamma_{\lambda,\mu} - \Gamma^{\nu}_{\lambda\mu}\gamma_{\nu} - [\Gamma_{\mu}, \gamma_{\lambda}] = 0 , \tag{2.5}$$

where $\Gamma^{\nu}_{\lambda\nu}$ is the Christoffel affinity

$$\Gamma^{\nu}_{\lambda\mu} = \tfrac{1}{2} g^{\nu\varrho}(g_{\lambda\varrho,\mu} + g_{\mu\varrho,\lambda} - g_{\lambda\mu,\varrho}) . \tag{2.6}$$

From the integrability condition

$$\gamma_{\lambda,\mu\nu} - \gamma_{\lambda,\nu\mu} = 0 , \tag{2.7}$$

it follows that the spinor tensor curvature $\mathfrak{R}_{\mu\nu}$,

$$\mathfrak{R}_{\mu\nu} = \Gamma_{\mu,\nu} + \Gamma_{\mu}\Gamma_{\nu} - \Gamma_{\nu,\mu} - \Gamma_{\nu}\Gamma_{\mu} , \tag{2.8}$$

is given in terms of the Riemann-Christoffel tensor $R^{\varrho}{}_{\lambda\mu\nu}$

$$R^{\varrho}{}_{\lambda\mu\nu} = \Gamma^{\varrho}{}_{\lambda\mu,\nu} + \Gamma^{\sigma}{}_{\lambda\mu}\Gamma^{\varrho}{}_{\sigma\nu} - \Gamma^{\varrho}{}_{\lambda\nu,\mu} - \Gamma^{\sigma}{}_{\lambda\nu}\Gamma^{\varrho}{}_{\sigma\mu} , \tag{2.9}$$

according to the formula

$$[\gamma_{\lambda}, \mathfrak{R}_{\mu\nu}] = R^{\varrho}{}_{\lambda\mu\nu}\gamma_{\varrho} . \tag{2.10}$$

One requires that $\mathfrak{R}_{\mu\nu}$ be a member of the γ_{μ} ring, so that

$$\mathfrak{R}_{\mu\nu} = -\tfrac{1}{4} R_{\varrho\lambda\mu\nu}\gamma^{\delta}\gamma^{\lambda} - ieF_{\mu\nu} . \tag{2.11}$$

On noting that for a Dirac spinor

$$\psi_{;[\mu\nu]} = \psi_{;\mu;\nu} - \psi_{;\nu;\mu} = -\mathfrak{R}_{\mu\nu}\psi , \tag{2.12}$$

one is led to the usual identification of (the real part of) $eF_{\mu\nu}$ as the product of the electromagnetic field and the particle charge. The measure of arbitrariness implicit in this identification is well brought out when one recasts the preceding discussion in terms of the generalized Duffin-Kemmer (9) matrices, to find that in the equation that replaces (2.11) the terms $eF_{\mu\nu}$ for inequivalent irreducible representations are unrelated, so that a spin-one particle might « see » a different electromagnetic field from that « seen » by a spin-zero par-

(9) R. J. Duffin: *Phys. Rev.*, **54**, 114 (1938); N. Kemmer: *Proc. Roy. Soc.*, A **173**, 91 (1939).

ticle. The spector that we use in subsequent Sections for the description of spin-$\frac{3}{2}$ fermions is the generalization of the Rarita-Schwinger wave function, ψ_μ, which has the notable property that

$$\psi_{\lambda;[\mu\nu]} = -\mathfrak{R}_{\mu\nu}\psi_\lambda - R^\varrho{}_{\lambda\mu\nu}\psi_\varrho . \tag{2.13}$$

We complete this Section by writing down for future reference some algebraic formulae

$$\gamma^\mu\gamma^\lambda\gamma^\nu = \gamma^\nu\gamma^\lambda\gamma^\mu + 2g^{\mu\lambda}\gamma^\nu - 2g^{\lambda\nu}\gamma^\mu , \tag{2.14}$$

$$R^\varrho{}_{\lambda\mu\nu} + R^\varrho{}_{\mu\nu\lambda} + R^\varrho{}_{\nu\lambda\mu} = 0 . \tag{2.15}$$

Equations (2.14) and (2.15) imply

$$R^\varrho{}_{\lambda\mu\nu}\gamma^\lambda\gamma^\mu\gamma^\nu = -2R^\varrho{}_{\lambda\mu\nu}g^{\nu\lambda}\gamma^\mu = -2R^\varrho{}_\mu\gamma^\mu , \tag{2.16}$$

from which it follows that

$$R_{\alpha\beta\mu\nu}\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu = -2R_{\alpha\beta\mu\nu}g^{\alpha\mu}g^{\beta\nu} = -2R . \tag{2.17}$$

3. – On the R.S. equation.

The aim of this Section is to examine the appropriateness of the Rarita-Schwinger ([3]) (R.S.) spector wave equations for the description of particles of spin $\frac{3}{2}$ in a universe endowed with a Riemannian metric. We take the R.S. equations in the form involving the « minimal extension » of the flat space equations, interpreting spectors as spectors in general relativity, and replacing the ordinary derivative by the covariant derivative,

$$\partial_\mu \to D_\mu \qquad \text{or} \qquad (\;)_{,} \to (\;)_{;} . \tag{3.1}$$

One remarks as an aside that this notation for covariant derivatives leads to no ambiguities as

$$D_\mu\gamma_\lambda - \gamma_\lambda D_\mu = \gamma_{\lambda;\mu} = 0 . \tag{3.2}$$

The R.S. equations we are to examine are thus simply

$$(\gamma^\mu D_\mu + m)\psi_\lambda = 0 , \tag{3.3}$$

9066

together with the supplementary condition

$$\gamma^{\lambda}\psi_{\lambda}=0\,. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$D^{\lambda}\psi_{\lambda}=0\,. \tag{3.5}$$

Thus, were ψ_{λ} subject to no further constraints, one would deduce that it had an equal number of independent components in any space, and so served to describe spin $\frac{3}{2}$. However, such is not the case, for we can combine (3.4) with (3.3) to produce an additional constraint equation. Starting from the equation (3.3) one writes down

$$\gamma^{\lambda}(\gamma^{\mu}D_{\mu}-m)(\gamma^{\nu}D_{\nu}+m)\psi_{\lambda}=0 \tag{3.6}$$

and uses (3.4) and (2.13) to deduce

$$\gamma^{\lambda}\gamma^{\mu}\gamma^{\nu}[\mathfrak{R}_{\mu\nu}\psi_{\lambda}+R^{\varrho}{}_{\lambda\mu\nu}\psi_{\varrho}]=0\,, \tag{3.7}$$

where $\mathfrak{R}_{\mu\nu}$ is given by (2.11). Using formulas (2.14), (2.16), (2.17) this last relation becomes

$$\tfrac{1}{2}R\gamma^{\lambda}\psi_{\lambda}-ieF_{\mu\nu}(\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}+4g^{\mu\lambda}\gamma^{\nu})\psi_{\lambda}-2R^{\mu\lambda}\gamma_{\mu}\psi_{\lambda}=0\,, \tag{3.8}$$

whence we determine the new constraint

$$(R^{\mu\lambda}-2ieF^{\mu\lambda})\gamma_{\mu}\psi_{\lambda}=0\,. \tag{3.9}$$

Tnis new constraint must be read as a constraint on the gravitational and electromagnetic fields, otherwise ψ_{λ} has insufficient degrees of freedom. Hence, factoring out the field,

$$R_{\mu\nu}=\lambda(x)g_{\mu\nu} \tag{3.10}$$

and

$$eF_{\mu\nu}=0\,. \tag{3.11}$$

By reference to the Bianchi identity eq. (5.12) below, it is seen that $\lambda(x)=\lambda$ is in fact a constant. Equation (3.10) is the field equation for the gravitational field when the matter tensor vanishes.

9067

We conclude that the Rarita-Schwinger equation together with the supplementary condition will serve to describe a spin-$\frac{3}{2}$ fermion in a passive gravitational field where the interactions of the fermion on the gravitational field and on the electromagnetic field are neglected.

The conclusions of this Section are not essentially novel. In the context of special relativity FIERZ and PAULI ([2]) pointed out that the Dirac-Fierz-Pauli (D.F.P.) equations for particles of spin $S \geqslant \frac{3}{2}$ impose an excessive number of constraints on the 2-component spinor wave functions if a minimal electromagnetic interaction is introduced by the replacement $\partial_\mu \to \partial_\mu - ieA_\mu$, $F_{\mu\nu} \neq 0$. BUCHDAHL ([4]) has shown that, likewise, the D.F.P. equations for $S \geqslant \frac{3}{2}$ when generalized in a minimal manner to an arbitrary Riemann space, are satisfactory if and only if the space is of constant Riemann curvature.

4. – A covariant wave equation for $S = \frac{3}{2}$.

Rather than postulating separately a wave equation and a supplementary condition whose compatibility would need to be examined, we seek a single equation which will *entail* appropriate supplementary conditions. (In like manner the Proca-Maxwell equation for massive spin-one bosons *entails* the supplementary condition $\partial_\mu \varphi^\mu = 0$). On taking into consideration the properties of the γ_μ matrices detailed above, it will be seen that the most general covariant equation for the R.S. spector ψ_μ which is of first order in the covariant derivative and free of terms involving the curvature tensor, the alternating tensor, or the electromagnetic field is

$$(\mathfrak{A}^\varrho{}_{\nu\mu} D_\varrho + m\mathfrak{B}_{\mu\nu})\psi^\nu = 0\,, \tag{4.1}$$

where

$$\mathfrak{A}^\varrho{}_{\mu\nu} = \gamma^\varrho g_{\mu\nu} + A(\delta^\varrho{}_\mu \gamma_\nu + \delta^\varrho{}_\nu \gamma_\mu) + B\gamma_\mu \gamma^\varrho \gamma_\nu \tag{4.2}$$

and

$$\mathfrak{B}_{\mu\nu} = g_{\mu\nu} + C\gamma_\mu\gamma_\nu\,; \tag{4.3}$$

A, B, C are scalars which for simplicity we restrict to constant values; likewise m is taken to be constant, although its re-interpretation as a variable mass field seems most plausible in a general relativity context ([8]).

It is easy to verify on contracting (4.1) with both D^μ and γ^μ and comparing coefficients that the term in $D^\mu\psi_\mu$ can be eliminated from (4.1), and a supplementary condition for $\gamma^{\mu}\psi_{\mu}$ can be deduced if (*)

$$A \neq -\tfrac{1}{2}\,, \qquad B = \tfrac{3}{2}A^2 + A + \tfrac{1}{2}\,, \qquad C = -3A^2 - 3A - 1\,. \tag{4.4}$$

Using these values, one sees upon contracting (4.1) with γ^{μ} that

$$D^{\mu}\psi_{\mu} = -\tfrac{1}{2}[(3A+1)\gamma^{\mu}D_{\mu} - 3(2A+1)m]\gamma^{\nu}\psi_{\nu}\,. \tag{4.5}$$

Contracting (4.1) with

$$[(A+1)\gamma^{\varrho}D_{\varrho} + m]\gamma^{\mu} - 2(2A+1)D^{\mu}\,, \tag{4.6}$$

it follows at once that

$$3(A+\tfrac{1}{2})^2 m^2 \gamma^{\nu}\psi_{\nu} = -(A+\tfrac{1}{2})(g^{\lambda\nu}\gamma^{\mu} + \tfrac{1}{2}A\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda})\psi_{\lambda;[\mu\nu]}\,. \tag{4.7}$$

This last relation may be reduced using the identities written down in Sect. 2 to give the following supplementary condition consistent with (4.1), and clearly embodying just as many constraints on ψ_{λ} in any space:

$$[3(A+\tfrac{1}{2})m^2 - \tfrac{1}{4}AR + \tfrac{1}{2}A\gamma^{\mu}F_{\mu\nu}\gamma^{\nu}]\gamma^{\nu}\psi_{\nu} = (\tfrac{1}{2}R^{\mu\nu} - ieF^{\mu\nu})\gamma_{\mu}\psi_{\nu}\,. \tag{4.8}$$

The question that does arise at once is what is the significance of the constant A. Now, under the transformation

$$\psi_{\mu} \to \psi_{\mu} + \tfrac{1}{4}\alpha\gamma_{\mu}\gamma^{\nu}\psi_{\nu}\,, \tag{4.9}$$

where $\alpha \neq -1$, $\alpha \neq -\tfrac{1}{2}$, our eq. (4.1) is form invariant, but

$$A \to [(2\alpha+1)A + \tfrac{1}{2}\alpha]/(\alpha+1)\,. \tag{4.10}$$

Clearly this transformation is the analogue of a gauge transformation for the Maxwell field; and in flat space one can readily show ([6]) that this transformation serves to mix the two classes of spin-$\tfrac{1}{2}$ fields contained in the spector ψ_{μ}; thus A has no physical significance, and may arbitrarily be assigned any value (provided $A \neq -\tfrac{1}{2}$). This conclusion is confirmed by the analysis of Moldauer and Case ([5]) which demonstrated that (in flat space) the magnetic moment and the electric quadrupole moment of the spin-$\tfrac{3}{2}$ fermion are independent of A.

Henceforth we take $A = -1$. Eliminating from eq. (4.1) the term in $D^{\mu}\psi_{\mu}$

(*) There is a misprint in the corresponding value of B given in ref. ([5]).

by substitution therein of (4.5) gives

$$(4.11)\qquad [(\gamma^\varrho D_\varrho + m)g^{\mu\nu} - D^\mu\gamma^\nu + \tfrac{1}{2}m\gamma^\mu\gamma^\nu]\psi_\nu = 0\,,$$

which is free of terms $m\,\gamma^\mu\gamma^\lambda D_\lambda\gamma^\nu\psi_\nu$. On contracting this equation with γ_μ one deduces the subsidiary condition (4.5) for $A = 1$. The other subsidiary condition may be deduced by contracting (4.1) with

$$(4.12)\qquad D_\mu + 2m\gamma_\mu$$

the result being of an especially attractive form, *viz.*

$$(4.13)\qquad 3m^{2\gamma\mu}\psi_\mu = -\gamma_\mu(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R^\alpha{}_\alpha)\psi_\nu - ie\gamma^\varrho F_{\varrho\mu}(\gamma^\mu\gamma^\nu - \tfrac{1}{2}g^{\mu\nu}\gamma^\alpha\gamma_\alpha)\psi_\nu\,.$$

5. – A covariant wave equation for $S = 2$.

The problem of devising consistent (multi-) vector wave equations to describe charged bosons in an arbitrary gravitational field differs in two points of detail from the problem of establishing consistent spector wave equations for charged fermions. First, remembering that by the operator D_μ we mean the usual covariant derivative, then

$$(5.1)\qquad (D_\mu D_\nu - D_\nu D_\mu)\varphi_\lambda = -\varphi_{\lambda;[\mu\nu]} = R^\varrho{}_{\lambda\mu\nu}\varphi_\varrho$$

(where φ_λ is a vector), which is to be compared with eqs. (2.11), (2.12): the electromagnetic interaction for bosons does not enter so naturally, but must be introduced by the covariant extension

$$(5.2)\qquad \partial_\mu \to \Pi_\mu = D - ieA_\mu\,.$$

The second difference comes about because the boson wave equations are of second order in the derivative, so that the covariant extension is undetermined by a constant factor $\varkappa$

$$(5.3)\qquad \partial_\mu\partial_\nu \to (1-\varkappa)\Pi_\mu\Pi_\nu + \varkappa\Pi_\nu\Pi_\mu = \Pi_\mu\Pi_\nu + \varkappa(\Pi_\nu\Pi_\mu - \Pi_\mu\Pi_\nu)\,.$$

Thus arguments based on the concept of « minimal electromagnetic coupling » cannot strictly eliminate these commutator terms. For spin one the covariant extension of the Proca-Maxwell equation is

$$(5.4)\qquad \Pi^\mu\Pi_\mu\varphi_\nu - \Pi^\mu\Pi_\nu\varphi - m^2\varphi_\nu - ie\varkappa F_{\mu\nu} + \varkappa R_{\mu\nu}\varphi^\mu = 0\,,$$

which implies the subsidiary condition

$$m^2\Pi^\nu\varphi_\nu = ie(1+\varkappa)F^{\mu\nu}\Pi_\mu\varphi_\nu + ie\varkappa F^{\mu\nu}{}_{;\mu}\varphi_\nu + \varkappa\Pi_\mu R^{\mu\nu}\varphi_\nu \,. \tag{5.5}$$

In deriving this constraint one uses the following identity for a tensor $T_{\mu\nu}$:

$$(D^\mu D^\nu - D^\nu D^\mu)T_{\mu\nu} = 0 \,. \tag{5.6}$$

However for the sake of avoiding algebraic complications we have omitted terms in a $\varkappa$-parameter in the spin-two equation. The wave function to describe a spin-two boson is taken to be a symmetric tensor

$$\varphi_{\mu\nu} = \varphi_{\lambda\mu} \,. \tag{5.7}$$

In applying a discussion akin to that used for spin $\frac{3}{2}$ to fix a wave equation, one must ensure that the wave equation is symmetric in the two free vector indices, otherwise additional supplementary constraints will be introduced. The wave equation is then found to be free of parameters

$$\begin{aligned}(\Pi^2 - m^2)\varphi_{\mu\lambda} - \Pi_\mu\Pi^\nu\varphi_{\nu\lambda} - \Pi_\lambda\Pi\varphi_{\nu\mu} + \\ + \tfrac{1}{3}g_{\mu\lambda}\Pi^\varrho\Pi^\sigma\varphi_{\sigma\varrho} + \tfrac{1}{3}m^2 g_{\mu\lambda}g^{\varrho\sigma}\varphi_{\sigma\varrho} - \tfrac{1}{3}g_{\mu\lambda}\Pi^2 g^{\varrho\sigma}\varphi_{\sigma\varrho} + \tfrac{1}{3}\Pi_\mu\Pi_\lambda g^{\varrho\sigma}\varphi_{\sigma\varrho} = 0 \,.\end{aligned} \tag{5.8}$$

On contracting (5.8) with

$$\Pi^\mu\Pi^\lambda - \Pi^2 g^{\mu\lambda} \,, \tag{5.9}$$

one deduces

$$m^2 g^{\mu\lambda}\varphi_{\mu\lambda} = 2\Pi^\varrho\Pi^\sigma\varphi_{\sigma\varrho} \,. \tag{5.10}$$

The other supplementary condition is deduced on contraction with Π_μ using (5.10) to find that

$$m^2\Pi^\mu\varphi_{\mu\lambda} = [\Pi^\mu, \Pi^2]\varphi_{\mu\lambda} + [\Pi^2, \Pi_\lambda]g^{\varrho\sigma}\varphi_{\sigma\varrho} \,. \tag{5.11}$$

In the absence of an electromagnetic interaction ($eF_{\mu\nu} = 0$) eq. (5.11) may be reduced using the well-known Bianchi identity ([10])

$$R^{\varrho\sigma\nu\mu}{}_{;\nu} = R^{\mu\varrho;\sigma} - R^{\mu\sigma;\varrho} \tag{5.12}$$

([10]) See, *e.g.*, C. Møller: *The Theory of Relativity* (Oxford, 1952).

to the simpler form

$$(5.13)\qquad m^2 D^\mu \varphi_{\mu\lambda} = R^{\mu\nu} D_\nu \varphi_{\mu\lambda} - 2R^{\mu\varrho\nu}{}_\lambda D_\varrho \varphi_{\mu\nu} + \\ + R_{\lambda\nu} D^\nu g^{\varrho\sigma} \varphi_{\varrho\sigma} + g^{\mu\nu} R_{;\mu} \varphi_{\nu\lambda} + R_{\mu\lambda;\nu} \varphi^{\mu\nu} - R_{\mu\nu;\lambda} \varphi^{\mu\nu} .$$

On the other hand, in flat space one has

$$(5.14)\qquad m^2 \Pi^\mu \varphi_{\mu\lambda} = -ie(F^{\alpha\beta}\Pi_\beta + \Pi_\beta F^{\alpha\beta})(\varphi_{\alpha\lambda} - g_{\alpha\lambda} g^{\mu\nu} \varphi_{\mu\nu}) .$$

It is interesting to compare our account of charged spin two in flat space with that given by FIERZ and PAULI (2), who also utilise a symmetric tensor $\varphi'_{\mu\nu}$ but impose the constraint

$$(5.15)\qquad g^{\mu\lambda} \varphi'_{\mu\lambda} = 0 ,$$

so that to achieve consistent equations a scalar field C must be introduced. An equation purporting to describe uncharged particles of spin two has been devised and examined by BUCHDAHL (4). However our general equation which can be specialized to these two limit situations, has moreover been devised by the same procedure that was applicable to the spin-$\frac{3}{2}$ problem; barring unforseen difficulties we anticipate that the procedure will prove adequate to specify wave equations to describe particles of any spin.

* * *

I would like to thank Prof. H. A. BUCHDAHL for directing my attention to this problem, and I acknowledge a useful discussion with Prof. H. S. GREEN.

RIASSUNTO (*)

È noto che sorgono particolari difficoltà quando si vuole impostare un'equazione d'onda che descriva particelle di spin superiore o uguale a $\frac{3}{2}$ interagenti con un campo elettromagnetico, e Buchdahl ha mostrato come simili difficoltà sorgano anche quando si introduce un campo gravitazionale. In questo articolo si fa vedere che come si può introdurre una interazione elettromagnetica coerente, così si possano trovare equazioni d'onda descriventi particelle cariche di spin $S=\frac{3}{2}$ o $S=2$ in un universo dotato di metrica Riemanniana. Nell'articolo si includono considerazioni sulle matrici γ di Dirac in relatività generale.

(*) *Traduzione a cura della Redazione.*

Ковариантные волновые уравнения
для заряженных частиц с высшими спинами в произвольном гравитационном поле.

Резюме (*). — Известно, что особые трудности встречаются при выводе волнового уравнения для описания частиц с высшими спинами $S \geqslant \frac{3}{2}$, взаимодействующих с электромагнитным полем, и Букдахл показал, что аналогичные трудности возникают, когда вводится гравитационное поле. Мы показываем, когда может быть введено соответствующее электромагнитное взаимодействие, также могут быть выведены волновые уравнения, описывающие заряженные частицы со спинами $S=\frac{3}{2}$ и $S=2$ во Вселенной, имеющей риманову метрику. Статья включает описание матриц Дирака γ в общей теории относительности.

(*) *Переведено редакцией.*

H. A. COHEN
21 Dicembre 1967
Il Nuovo Cimento
Serie X, Vol. 52 A, pag. 1242-1253